

The University of Chicago

A SEMI-ANALYTICAL
METHOD OF FACTORIAL ROTATION
TO SIMPLE STRUCTURE

A DISSERTATION SUBMITTED TO THE FACULTY
OF THE DIVISION OF THE BIOLOGICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF DOCTOR OF
PHILOSOPHY

DEPARTMENT OF PSYCHOLOGY
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A factorial rotational method is presented which represents a compromise between the use of subjective judgment characteristic of graphical methods and the routine application of analytical methods. At present the analytical methods seem to be inadequate for the discovery of a simple structure, while graphical methods require more subjective judgment. The method herein presented locates the axes for subgroups of tests by an analytical method. The judgments used in the selection of subgroups are based on graphic data concerning interrelation of the factors.

During the development of multiple factor analysis the rotation of the reference axes has risen to great importance. The position of the accepted set of axes depends upon the purpose of the individual factorial study, as does also the method of rotation employed. In general, that set of axes should be selected which is most useful in solving the problem for which the study was devised. Thus the principle underlying the method of rotation depends upon the use to which the entire study is to be put. For some problems the principal axes are the most useful; other studies merely demand that the axes be orthogonal; still others require rotation to positions where the underlying nature of the factors is more easily deduced from the saturations of the tests. It was for this last case that Thurstone (3) developed the principle of simple structure. It is the belief of those who employ this principle that axes can be passed through the factorial configuration so that a large number of tests have vanishingly small loadings.

Although the principle of simple structure seems to be clear-cut, the methods of rotation have been cumbersome and difficult to apply. However, sufficient experience with these methods has now been obtained to allow a critical review of their difficulties as well as of their merits and thus to permit the development of a compromise between the routinely applied analytical methods and the more subjective graphical methods.

The most successful rotational methods have been the graphical ones devised by Thurstone (5, 6). The results from these procedures

have, in general, been satisfactory; however, these methods depend upon subjective judgments of the investigator and uniform results are not obtained by all. Successful solutions of rotational problems by these methods depend upon a highly sophisticated sense of geometrical projection and of fitting lines to points. Further, the computational labor seems unnecessarily great, especially when the results are carried to a high degree of precision.

In terms of objectivity of procedure an analytical method of rotation is to be preferred. Identical results could then be obtained by all investigators, and thus the solution could be termed unique. However, analytical procedures have a number of inherent difficulties. Usually, for each test j a value y_j is assigned for the corresponding loading v_j and then the sum of the y 's is maximized, or minimized, for each of the rotated axes. Whether a maximum solution or a minimum solution is desired depends upon the definition of the y 's. An example of a case where a maximum sum of y 's is desired is when

$$y_j = e^{-cv_j^2},$$

and

$$\sum_j y_j = \text{a maximum.}$$

In this case y_j increases when the absolute value of v_j decreases and has its highest value for a v_j of zero. A minimum sum of y 's is desired when the lowest value of y_j corresponds to a zero v_j . The important point to note is that all tests are entered undifferentially into the formulation. Thus the position of the axis is influenced not only by those tests which are to have zero or near zero loadings, but also by those with high loadings. This results in a shift in the axis from the desired position. The amount of this shift varies both with the study and with the particular definition of y_j . An exceptionally bad example is seen in the case where y_j is defined as the square of v_j . When the sum of the y 's is minimized the result is the mean principal axis of the entire system where all of the test projections are brought to as near zero as possible. Many of the other definitions of y_j tend toward this same result.

Thurstone (4) has suggested, as a possible means of overcoming the inadequacies of analytical criteria, the consideration of a subgroup of tests rather than the entire battery when taking the sum of the y 's. This procedure was further developed by Tucker (7), who defined the simple structure factor as the mean principal axis of the zero subgroup of tests. In general, however, the solutions for any assumed subgroup cannot be easily obtained directly and recourse has been made to laborious methods of successive approximation. An-

other difficulty is in the selection of the subgroup to be treated. If the subgroup is taken as those tests which lie within a certain range from zero on a trial vector, the resulting solution is merely a tightening up of the tests in this range and the trial vector could easily remain a combination of two or three of the simple structure factors. This results because all tests high on the two or three factors have loadings on the original trial vector outside of the range taken in defining the subgroup. A more satisfactory method is to solve for all factors simultaneously, selecting subgroups from the inter-factor graphs.

Although the use of subgroups has led to improved results, there still exists a small tendency for the trial factors to rotate toward the mean principal axis of the entire system. This tendency is especially pronounced when the variance on the mean principal axis is low and this axis can be considered as a residual dimension. The major defect of rotation in this direction is that the tests with non-vanishing loadings on the factors have smaller loadings than if this rotation were not carried out. There is a loss of variance of projection for these tests.

A solution for this problem of loss in size of the high projections is possible with the criterion suggested by Horst (1) who maximized the ratio of the sum of squares of the high projections to the sum of squares of the low projections; that is, he divided the tests into two subgroups, one for those tests with non-vanishing loadings and one for those tests whose loadings were to be vanishing. The tests in the high subgroup will be designated here by the subscript h , those in the zero subgroup by the subscript z . Horst's solution was then to make

$$\frac{\sum_h v_h^2}{\sum_z v_z^2} = \text{a maximum.} \quad (1)$$

This criterion seems to have the proper characteristics when the correct subgroup has been chosen. However, in applying this criterion Horst did not make use of graphical selection of the subgroups, preferring, rather, the use of a range of projections on a previous trial. The method proposed here makes the solution for all factors simultaneously, using the inter-factor graphs as guides for the selection of the subgroups. As a further aid in computation, an approximation to Horst's criterion is used during the trials where the subgroups are being selected.

The fictitious problem used by Thurstone to illustrate rotation with extended vectors (6) will be used here to demonstrate the prop-

erties of the proposed method. Table 1 presents the fictitious simple configuration which was rotated to the centroid matrix of Table 2. In practice the problem is to start with a centroid matrix such as is given in Table 2 and rotate to the simple configuration. In this case, however, it is convenient to have the exact solution given in Table 1 in order to judge the goodness of the rotations.

The first problem in applying the method is that of obtaining an initial set of trial subgroups. These subgroups may be obtained by a cluster analysis of the original correlations, by inspection of the centroid factorial matrix, or, as is suggested here, by a cluster analysis of the first factor residuals. It is assumed that the configuration has not only a simple structure, but also a positive manifold. When this is true, as with most test data, the first centroid axis passes somewhere in the middle of the positive manifold and the test vectors are projected down into the space orthogonal to this axis. Not all the residual correlations are positive. In fact, positive residuals occur only for those pairs of tests that extend in the same general directions and thus have loadings on the same factors.

Table 3 gives the first factor residuals for the illustrative problem. It is noted that the highest residual in the table is between variables 2 and 14. These two tests are taken as the starting point for the first cluster. The next test added to this cluster is test 4, which correlates .45 and .30 with tests 2 and 14, respectively. Succeeding tests are added to the subgroup in such a way that all of the residual intercorrelations are positive. The tests in this cluster are listed in column A of Table 4.

Cluster A accounts for a number of the positive residuals. The next cluster is started by finding the highest residual that has not been accounted for by cluster A. This residual is between variables 1 and 13. Cluster B, of Table 4, is found starting with these tests. All of the residual intercorrelations within this cluster are also positive. The other clusters are found in a similar manner. Since the problem has five dimensions as shown by the five columns of the centroid matrix of Table 2, five clusters are taken. In general, as many clusters should be taken as there are dimensions in the factorial matrix. These clusters will be used as the first assumed subgroups of "high" tests.

There is one other preparatory set of computations to be made before starting on Trial 1. The centroid factorial matrix of Table 2 must be premultiplied by its transpose, thus producing the matrix P ; that is,

$$P = F'_0 F_0. \quad (2)$$

This product matrix for the illustrative problem is given in Table 5. The diagonal entries are the sums of the squares of the corresponding columns of F_0 and the off-diagonal entries are the sums of cross-products between columns of F_0 . The inverse of P is also found and is listed in Table 6. P^{-1} will be used in the calculations for each trial.

The computational work for Trial 1 is shown in Table 7. The matrix W has a row for each of the tests and a column for each factor. The factor A in matrix W corresponds to cluster A of Table 4. Each of the tests in this cluster has an entry of unity for this factor. All other cells in the column are left blank to indicate entries of zero. The other columns of W correspond to the other clusters.

The first step in Trial 1 is to find the matrix B by the equation:

$$B = F_0' W. \quad (3)$$

The matrix L is then found by:

$$L = P^{-1} B. \quad (4)$$

The columns of L are then normalized to the corresponding columns of Λ . The matrix C is found by:

$$C = \Lambda' \Lambda. \quad (5)$$

This matrix shows the cosines of the angles between the trial factors and should be closely inspected to see if any pair of the factors are nearly identical. If this should occur, a new cluster should be tried. This is shown by a high positive side entry. The diagonal entries should be unity if the normalizing has been done properly. The last computational step of the trial is the finding of the matrix of test projections V by the equation:

$$V = F_0 \Lambda. \quad (6)$$

When the matrix V has been obtained, a plot is made for each pair of columns. In the present example this results in ten graphs as shown in Figures 1 and 2. The foregoing procedure is followed for all trials except that the entries in the W matrix are derived from the diagrams rather than from the table of first factor residuals.

The entries in the W matrix of Trial 2 shown in Table 8 were derived from the graphs for Trial 1 shown in Figures 1 and 2. The graphs of factor A with the other factors were inspected in setting up the column A of W . The points 2, 14, 4, 3, and 23 are plainly high on factor A in all four graphs involving A. On the graph A-D, points 6 and 8 clearly seem to deserve high loadings on A. The con-

figuration of points in the E direction on the $A-E$ graph seems too wide for all of the tests to have vanishing loadings on factor A and thus tests 12 and 24 are included in the high group on factor A . The case of tests 1 and 5 in the B direction on the $A-B$ graph is more ambiguous; however, it was decided to include them also in the high group on factor A . Tests 21 and 25 on the $A-C$ graph were left in the zero group on factor A . The tests placed in the high group have unities entered in the A column of the W matrix of Table 8. The cells for the tests in the zero group are left blank. Each of the other columns in W were obtained in the same way by inspecting the graphs for each of these factors. The selection of the subgroups is the high point in the procedure for Trial 2, since the computational procedure is identical in form to that outlined in the preceding paragraph for Trial 1.

When the computations for Trial 2 are completed and the graphs made, it is found that all of the subgroups are satisfactory. Two of these graphs are illustrated in Figure 3; the graph $A-E$ seems to be the poorest of the ten and $C-D$ the best. In order to tighten the points to the planes a third trial is made in which the entries in the W matrix of Table 9 for the high tests are taken as equal to the v 's of Trial 2 rounded off to one decimal. Again the cells for the zero group are left blank. The computations are again the same as for Trial 1. The results of this trial are very near the ideal solution indicated in the original simple configuration of Table 1 except that the columns are in a different order. The order of the columns is of no consequence since they can be renamed and their order changed to any order desired. The graphs of Figure 4 illustrate the tightness of fit that has been obtained. All that is necessary to tighten the fit of the planes still further is to continue to another trial with improved weights in the W matrix taken from the present set of loadings in the V matrix of Table 9. However, since this was merely a repetition of the procedure already presented, it was not carried out.

A profitable variation in the procedure after Trial 1 is presented in Table 10 where the work of Trial 2' is given. The difference between Trial 2 of Table 8 and Trial 2' is in the use of differential weights in the W matrix of Trial 2'. These weights are assigned to the high tests from inspection of the graphs for the first trial which are presented in Figures 1 and 2. This was done at the time that the tests were placed in the high subgroup. Since the weights depend upon subjective estimates of final size of loadings from these graphs, it is probably best not to use too complicated or extensive a set. Therefore, it was decided to restrict these weights to the digits 1, 2, or 3. Tests 2, 14, and 4 are plainly higher than all of the other tests on factor A and are, therefore, given weights of 3. Tests 3 and 23

are given weights of 2, as are tests 6 and 8. These latter two tests seem from the graph *A-D* to have disproportionately low loadings on Trial 1. The other four tests in the high group, tests 1, 5, 12, and 24, are given weights of unity. The results of this trial are closer to the exact solution than are those of Trial 2, as will readily be seen by inspection of the *V* matrices of the two trials.

In order to gauge further the amount of discrepancy of each trial from the exact solution, the differences between the loadings in each trial and those in the exact solution were found and frequency distributions of the absolute values of these errors made. These distributions are shown in Table 11. It will be noted that each of the trials in the series 1, 2, and 3 are improvements over the preceding trials. Trial 2', where subjective differential weights are used, is definitely better than Trial 2, being almost as good as Trial 3. The use of simple differential weights is recommended because of this gain. Further reassurance can be obtained from the fact that errors in these weights are not a too serious matter, for the goodness of results should not regress any further than if all of the tests were given unit weights as in Trial 2. It merely retards progress.

The advantage of using the graphs in selecting the subgroups can be illustrated by the initial selection of a high subgroup which includes tests high on two factors, *B* and *C* in this case. In Table 12 an alternate factor *B'* for Trial 1 is presented. It is to be imagined that this factor replaces the *B* factor in Table 7. The graphs between this new factor and the other four factors of Trial 1 are presented in Figure 5. The most profitable diagram for selecting the next high subgroup is *B'-C*, where there are no tests holding the *B* plane in the direction of *C*. A line for the new plane can be drawn on this diagram and tests selected which are farther from it. In this way the factors are guided to their proper solutions.

The computational procedure can be derived by either of two mathematical rationales. In the first of these the criterion is defined as:

$$y_{jp} \equiv (w_{jp} - k v_{jp}), \quad (7)$$

and

$$\sum_j y_{jp}^2 = \text{a minimum.} \quad (8)$$

The subscript *p* is used to designate rotated factor; thus *w_{jp}* is the entry for test *j* and factor *p* in the *W* matrix. *v_{jp}* is the resulting projection of test *j* on factor *p*. The *w_{jp}*'s are in this rationale to be thought of as proportional to a hypothetical set of loadings which might be obtained at the solution. The *y_{jp}*'s are the errors in fitting an actual set of loadings to these hypothetical ones. *k* is used so as to

consider only the relative order of magnitude of the v_{jp} 's. This solution has been derived by Mosier (2), where the w_{jp} 's were obtained from previous factorial studies. In this case they are derived from inspection of the configuration of the battery being investigated. Equation (8) is a least-squares statement for the errors y_{jp} .

A matrix equation for the v_{jp} 's is given in (6). In summation form this is:

$$v_{jp} = \sum_m a_{jm} \lambda_{mp} \quad (9)$$

where a_{jm} is the entry for test j and reference factor m in the original factor matrix F_0 . λ_{mp} is the direction cosine for rotated factor p from the reference factor m . In order to facilitate the derivation it is convenient to define:

$$u_{jp} \equiv k v_{jp}; \quad (10)$$

for, then

$$u_{jp} = k \sum_m a_{jm} \lambda_{mp},$$

$$u_{jp} = \sum_m a_{jm} k \lambda_{mp},$$

or

$$u_{jp} = \sum_m a_{jm} l_{mp}, \quad (11)$$

where

$$l_{mp} \equiv k \lambda_{mp}. \quad (12)$$

The l_{mp} 's are direction numbers and are mutually independent. Equation (7) then becomes

$$y_{jp} = w_{jp} - u_{jp}. \quad (13)$$

In order to find the minimum solution of equation (8), the partial derivatives with respect to the l_{mp} 's are found and equated to zero. That is:

$$\frac{\partial \sum_j y_{jp}^2}{\partial l_{mp}} = 2 \sum_j y_{jp} \frac{\partial y_{jp}}{\partial l_{mp}};$$

using equation (13),

$$\frac{\partial \sum_j y_{jp}^2}{\partial l_{mp}} = 2 \sum_j (w_{jp} - u_{jp}) \left(- \frac{\partial u_{jp}}{\partial l_{mp}} \right),$$

and by (11),

$$\frac{\partial \sum_j y_{jp}^2}{\partial l_{mp}} = 2 \sum_j (w_{jp} - \sum_M a_{jM} l_{Mp}) (-a_{jM}) = 0. \quad (14)$$

M is used as an alternative subscript for m . Equation (14) can be simplified to:

$$-\sum_j w_{jp} a_{jm} + \sum_j \sum_M a_{jM} l_{Mp} a_{jm} = 0,$$

or

$$\sum_j w_{jp} a_{jm} = \sum_j \sum_M a_{jM} l_{Mp} a_{jm},$$

or

$$\sum_j w_{jp} a_{jm} = \sum_M l_{Mp} \sum_j a_{jM} a_{jm}. \quad (15)$$

Equation (15) in matrix form is:

$$\mathbf{F}_0' \mathbf{W} = \mathbf{F}_0' \mathbf{F}_0 \mathbf{L},$$

or by equations (2) and (3), becomes:

$$\mathbf{B} = \mathbf{P} \mathbf{L},$$

or

$$\mathbf{P}^{-1} \mathbf{B} = \mathbf{L}. \quad (4)$$

Equation (4) is the solution for minimizing the discrepancies between a hypothetical set of values given in $\mathbf{W}$ and values proportional by columns to the entries in $\mathbf{V}$. The factors approach the desired rotated solution as the w_{jp} 's are brought closer to the final v_{jp} 's. The discrepancies of the tests in the zero subgroup are constantly minimized since their w 's are always set at zero.

The second rationale from which the computational procedure can be obtained is slightly more complicated than the first, but throws some additional light on the nature of the solution. This rationale starts from Horst's criterion in equation (1). If all of the tests are either h tests or z tests,

$$\sum_j v_{jp}^2 = \sum_h v_{hp}^2 + \sum_z v_{zp}^2; \quad (16)$$

then

$$\sum_z v_{zp}^2 = \sum_j v_{jp}^2 - \sum_h v_{hp}^2,$$

and

$$\frac{\sum_h v_{hp}^2}{\sum_z v_{zp}^2} = \frac{\sum_h v_{hp}^2}{\sum_j v_{jp}^2 - \sum_h v_{hp}^2},$$

or

$$\frac{\sum_h v^2_{hp}}{\sum_z v^2_{zp}} = \frac{1}{\frac{\sum_j v^2_{jp}}{\sum_h v^2_{hp}} - 1}. \quad (17)$$

It is evident that when

$$\frac{\sum_h v^2_{hp}}{\sum_z v^2_{zp}} = \text{a maximum},$$

$$\frac{\sum_j v^2_{jp}}{\sum_h v^2_{hp}} - 1 = \text{a minimum};$$

then

$$\frac{\sum_h v^2_{hp}}{\sum_j v^2_{jp}} = \text{a maximum}. \quad (18)$$

This last equation provides a simpler basis for the development of both the exact solution and the approximate ones.

As in the first rationale, it is easier here to deal with direction numbers, l_{mp} 's, and values, u_{jp} 's, proportional to the v_{jp} 's. Substituting equation (10) in equation (18) and cancelling the $(1/k^2)$'s in the numerator and denominator,

$$\frac{\sum_h u^2_{hp}}{\sum_j u^2_{jp}} = \text{a maximum}. \quad (19)$$

It is convenient now to define a condition placed upon the l_{mp} 's; that is, it is defined that:

$$\sum_j u^2_{jp} \equiv 1. \quad (20)$$

With this condition it is necessary only to make

$$\sum_h u^2_{hp} = \text{a maximum}. \quad (21)$$

However, instead of continuing to the exact solution, consider the following equation:

$$\sum_j u_{jp} w_{jp} = \text{a maximum}. \quad (22)$$

When the w_{jp} 's are zero for the z tests and equal to the u_{hp} 's for the h tests, this equation becomes the same as (21). However, when the w_{hp} 's are plus or minus one, the signs being taken as the same as the u_{hp} 's, equation (22) becomes

$$\sum_h |u_{hp}| = \text{a maximum.} \quad (23)$$

Thus, with proper selection of the w_{hp} 's, the weighted sum of the u_{jp} 's becomes either the sum of the squares of the u_{hp} 's or the sum of the absolute values of the u_{hp} 's. This latter squared is used as approximately proportional to the former. That is:

$$[\sum_h |u_{hp}|]^2 \simeq c \sum_h u_{hp}^2. \quad (24)$$

The solution of equation (22) under the condition of equation (20) is accomplished with the use of LaGrange multipliers γ_p . Thus

$$\frac{\partial \sum_j u_{jp} w_{jp}}{\partial l_{mp}} + \gamma_p \frac{\partial (\sum_j u_{jp}^2 - 1)}{\partial l_{mp}} = 0. \quad (25)$$

Then

$$\sum_j w_{jp} \frac{\partial u_{jp}}{\partial l_{mp}} + 2 \gamma_p \sum_j u_{jp} \frac{\partial u_{jp}}{\partial l_{mp}} = 0;$$

or, using equation (11)

$$\sum_j w_{jp} a_{jm} + 2 \gamma_p \sum_j \sum_M a_{jM} l_{Mp} a_{jm} = 0,$$

or

$$\sum_j w_{jp} a_{jm} = -2 \gamma_p \sum_M l_{Mp} \sum_j a_{jM} a_{jm}. \quad (26)$$

However, since each column of L is to be separately normalized, the multipliers $(-2 \gamma_p)$ can be dropped and equation (26) becomes identical with equation (15) and reduces to equation (4) in a similar manner.

Since equation (24) is usually a good type of approximation, it was expected that the use of w_{hp} 's equal to unity should give good results. This expectation is at least partially born out by the results illustrated in Trials 1 and 2 presented in Tables 7 and 8. Better approximations can be obtained by using differential weights as in Trial 2' in Table 10 since these w_{hp} 's approach closer to being proportional to the solution v_{hp} 's.

Although the method was illustrated with a problem having a positive manifold, it is not restricted to such a problem. Tests with negative loadings can be entered into the high group by using negative w_{hp} 's. The only portion of the procedure that is not generally applicable is in the selection of the first trial subgroups. Some other principle than the one used here may have to be used in the case where there are tests with significant negative factor loadings.

The author is indebted to Professor L. L. Thurstone for his many suggestions and criticisms during the development of this rotational method. The present study is issued as a report from the Psychometric Laboratory at the University of Chicago and has been made possible by research grants from the Social Science Research Committee and from the Carnegie Corporation of New York for the Psychometric Laboratory. A further acknowledgment is due to Mrs. Mary S. Carroll for her editorial assistance during the preparation of the manuscript.

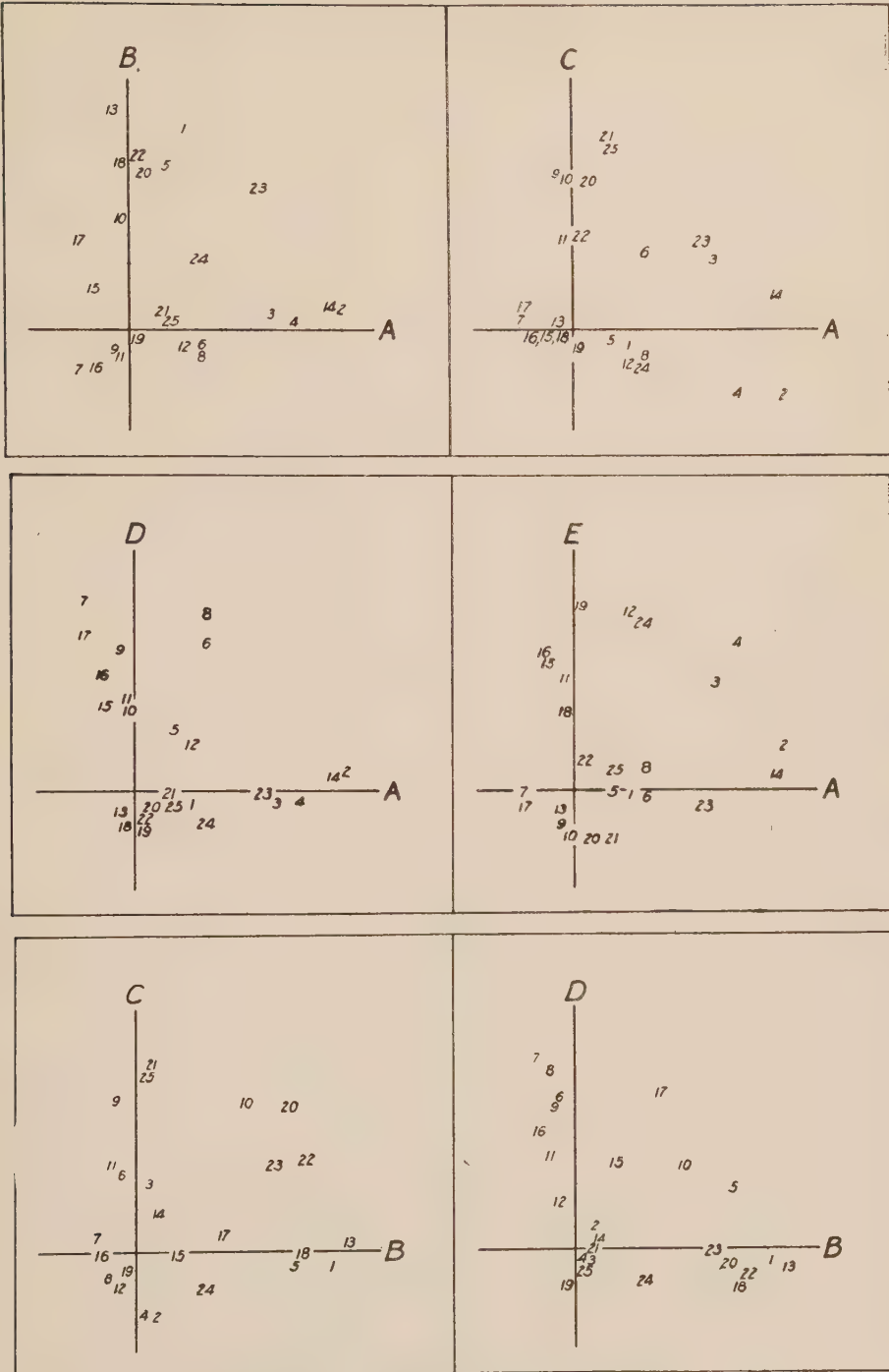


FIGURE 1

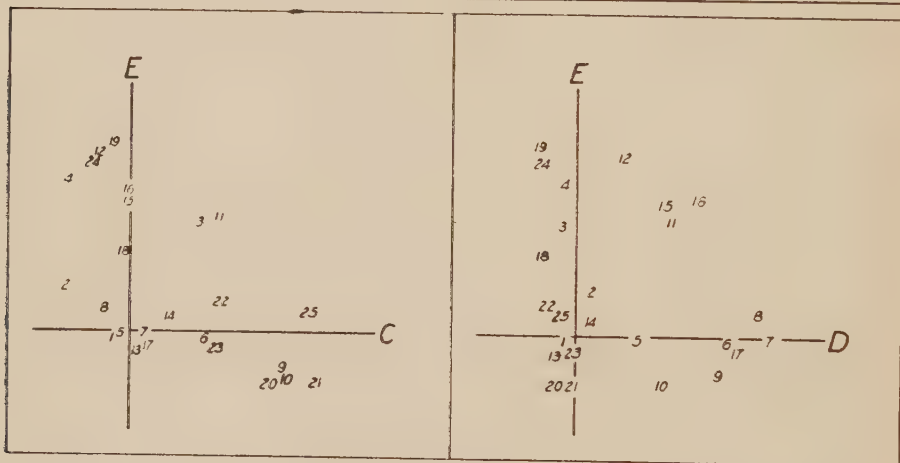
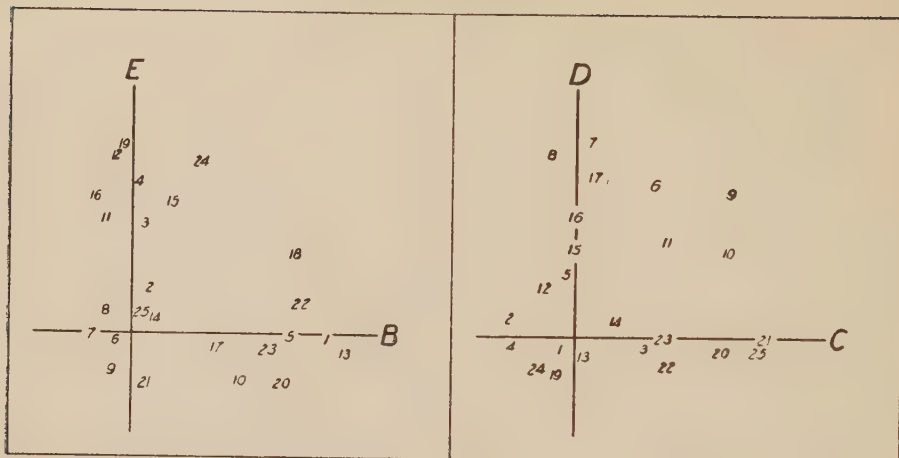


FIGURE 2

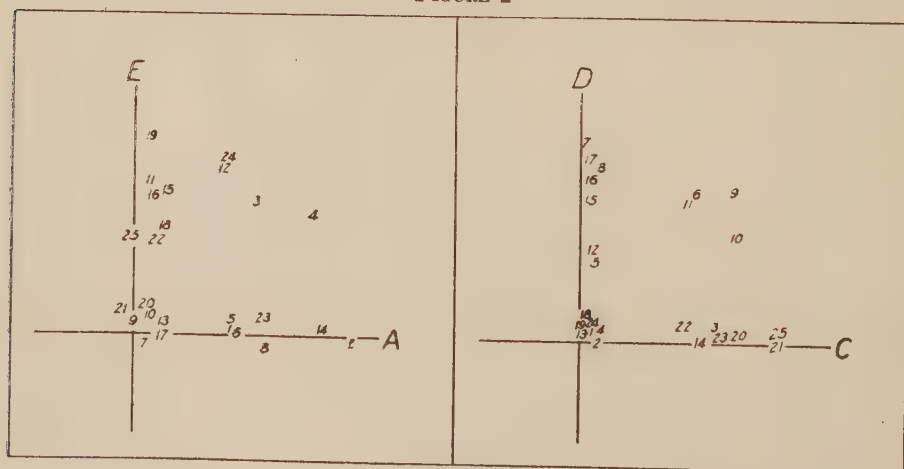


FIGURE 3

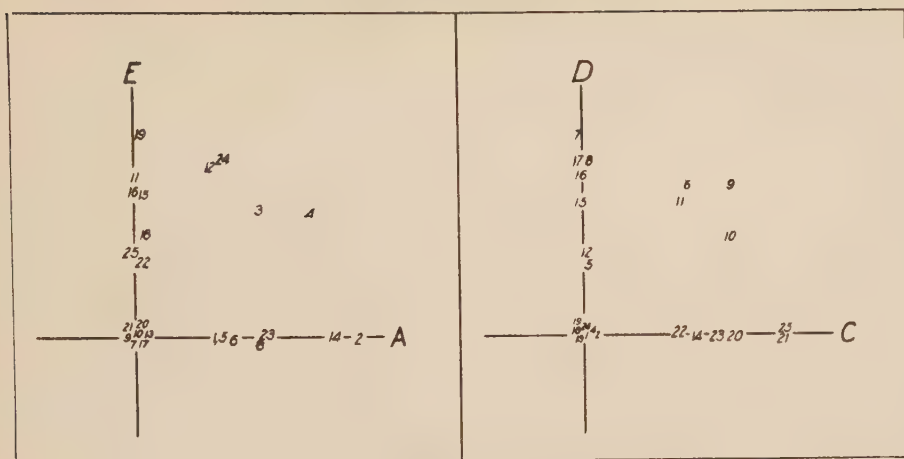


FIGURE 4

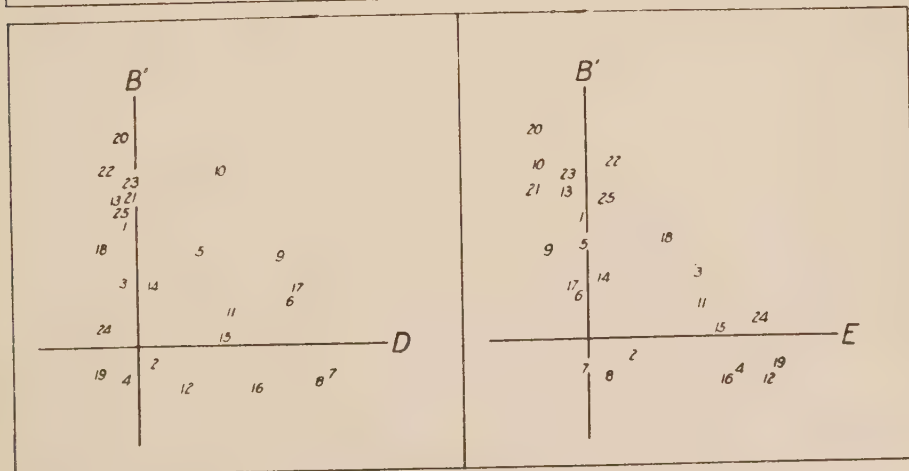
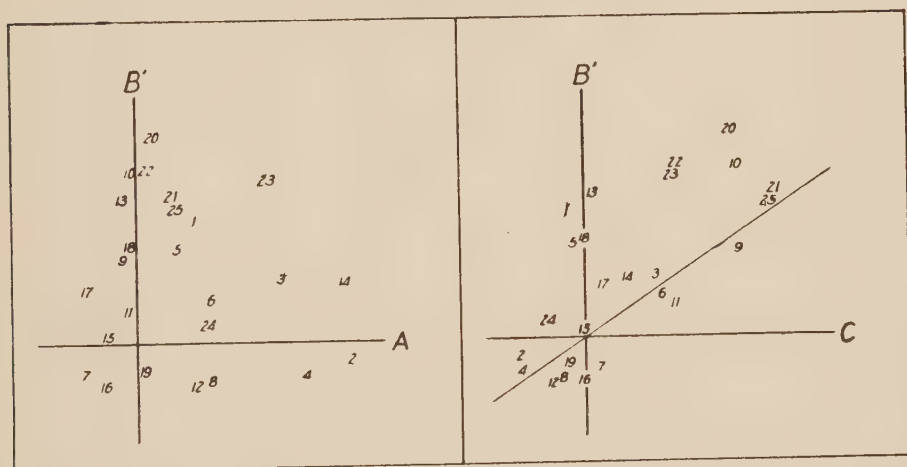


FIGURE 5

TABLE 1
A Fictitious Simple Configuration

	P_1	P_2	P_3	P_4	P_5
1	.0	.3	.0	.0	.8
2	.0	.9	.0	.0	.0
3	.5	.5	.0	.5	.0
4	.5	.7	.0	.0	.0
5	.0	.3	.3	.0	.7
6	.0	.4	.6	.4	.0
7	.0	.0	.8	.0	.0
8	.0	.5	.7	.0	.0
9	.0	.0	.6	.6	.0
10	.0	.0	.4	.6	.5
11	.6	.0	.5	.4	.0
12	.7	.3	.3	.0	.0
13	.0	.0	.0	.0	.9
14	.0	.8	.0	.4	.0
15	.6	.0	.5	.0	.3
16	.6	.0	.6	.0	.0
17	.0	.0	.7	.0	.5
18	.4	.0	.0	.0	.7
19	.8	.0	.0	.0	.0
20	.0	.0	.0	.6	.6
21	.0	.0	.0	.8	.0
22	.3	.0	.0	.4	.7
23	.0	.5	.0	.5	.5
24	.7	.3	.0	.0	.3
25	.3	.0	.0	.8	.0

TABLE 2
Centroid Factor Matrix, F_0

	I	II	III	IV	V
1	.51	— .19	.53	.33	— .22
2	.37	.49	.54	— .08	.38
3	.65	.47	— .03	— .30	— .08
4	.51	.31	.37	— .48	.15
5	.60	— .27	.37	.32	.02
6	.61	.18	— .21	.17	.45
7	.36	— .35	— .30	.10	.54
8	.52	— .03	.04	.05	.69
9	.54	.07	— .56	.26	.22
10	.69	— .06	— .26	.45	— .13
11	.67	— .08	— .46	— .31	.05
12	.57	— .07	.00	— .57	.13
13	.44	— .39	.39	.39	— .39
14	.51	.66	.25	.06	.21
15	.64	— .44	— .11	— .30	.04
16	.54	— .35	— .28	— .42	.24
17	.56	— .52	— .04	.31	.26
18	.52	— .37	.27	— .03	— .42
19	.36	— .12	— .07	— .67	— .23
20	.56	.07	— .07	.45	— .45
21	.36	.44	— .44	.25	— .25
22	.65	— .13	.06	.18	— .51
23	.67	.33	.24	.33	— .16
24	.58	— .07	.25	— .48	— .20
25	.49	.40	— .47	.00	— .33

TABLE 3
First Centroid Residuals *

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1		08	-19	-05	35	-19	-18	-11	-28	05	-35	-20	50	-02	-08	-28	12	30	-19	20	-18	23	21	03	-26
2	08		21	45	05	13	-14	27	-21	-26	-25	06	-16	54	-24	-20	-20	-19	-13	-21	-13	-24	20	06	-18
3	-19	21		27	-23	01	-23	-08	-04	-14	06	13	-28	27	-11	-05	-36	-14	16	-06	17	-08	06	12	23
4	-05	45	27		-10	-04	-18	08	-28	-35	-04	27	-22	30	-03	02	-28	-06	22	-29	-18	-18	01	27	-10
5	35	05	-23	-10		-06	02	05	-14	06	-25	-16	37	-06	-02	-14	23	18	-21	09	-21	10	10	-05	-29
6	-19	13	01	-04	-06		26	31	27	06	05	-05	-26	17	-09	03	08	-32	-22	10	10	-23	-01	-23	02
7	-18	-14	-23	-18	02	26		38	29	07	16	04	-15	-18	18	29	36	-18	-12	-20	-13	-23	-24	-20	-17
8	-11	27	-08	08	05	31	38		14	-08	00	06	-22	14	02	14	21	-27	-19	-29	-19	-34	-10	-15	-26
9	-28	-21	-04	-28	-14	27	29	14		23	18	-12	-23	-03	-04	08	12	-28	-19	06	29	-11	-06	-31	22
10	05	-26	-14	-35	06	06	07	-08	23		-02	-27	15	-10	-08	-12	14	-01	-24	27	23	14	09	-25	14
11	-35	-25	06	-04	-25	05	16	00	18	-02	19		-29	-17	18	30	-02	-11	24	-13	08	-10	-24	03	17
12	-20	06	13	27	-16	-05	04	06	-12	-27	19	-25		-05	20	29	-11	-01	36	-32	-20	-16	-23	25	-07
13	50	-16	-28	-22	37	-26	-15	-22	-23	15	-29	-25	-22		00	-23	20	40	-15	29	-15	34	15	02	-21
14	-02	54	27	30	-06	17	-18	14	-03	-10	-17	-05	-22	-33		-27	-28	-26	-18	-04	14	-17	26	-05	08
15	-08	-24	-11	-03	-02	-09	18	02	-04	-08	18	20	00	-33	32		15	12	25	-17	-23	-02	-28	14	-14
16	-28	-20	-05	02	-14	03	29	14	08	-12	30	29	-23	-27	32	32		-03	29	-30	-19	-17	-36	11	-09
17	12	-20	-36	-28	23	08	36	21	12	14	-02	-11	20	-28	15	12	12		06	-20	-01	-20	-12	-17	-28
18	30	-19	-14	-06	18	-32	-18	-27	-28	-01	-11	-01	40	-26	12	-03	06		14	13	-18	27	00	19	-14
19	-19	-13	16	22	-21	-22	-12	-19	-19	-24	24	36	-15	-18	25	-29	20	14		-20	-13	01	-24	36	06
20	20	-21	-06	-29	09	-10	-20	-29	06	27	-13	-32	29	-04	-17	-30	-01	13	-20		28	30	23	-15	21
21	-18	-13	17	-18	-21	10	-13	-19	29	23	08	-20	-15	14	-23	-19	-20	-18	-13	28		09	16	-21	47
22	23	-24	-08	-18	10	-23	-23	-34	-11	14	-10	-16	34	-17	-02	-17	-01	27	01	30	09		11	04	09
23	21	20	06	01	10	-01	-24	-10	-06	09	-24	-23	15	26	-28	-36	-12	00	-24	23	16	11		-09	07
24	03	06	12	27	-05	-23	-20	-15	-31	-24	03	25	02	-05	14	11	-17	19	36	-15	-21	04	-09		-08
25	-26	-18	23	-10	-29	02	-17	-26	22	14	17	-07	-21	08	-14	-09	-28	-14	06	21	47	09	07		-08

* The entries in this table were computed with the Matrix Multiplier (8) and may be in error by .01. All decimal points were omitted from this table; each entry has two decimal places.

TABLE 4
Positive Clusters of First Factor Residuals

<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>
2	1	21	7	12
14	13	25	8	19
4	18	9	17	24
3	5	20	6	4
23	22	10	9	16
	20		16	
	23			

TABLE 5
 $P = F_0' F_0$

	I	II	III	IV	V
I	7.5128	— .0857	— .0568	.0768	— .0940
II	— .0857	2.6564	— .0002	— .0707	.1038
III	— .0568	— .0002	2.5033	.0704	— .2887
IV	.0768	— .0707	.0704	2.8753	— .3891
V	— .0940	.1038	— .2887	— .3891	2.5289

TABLE 6
 P^{-1}

	I	II	III	IV	V
I	.1333	.0040	.0037	— .0029	.0048
II	.0040	.3773	— .0018	.0073	— .0144
III	.0037	— .0018	.4049	— .0039	.0458
IV	— .0029	.0073	— .0039	.3554	.0538
V	.0048	— .0144	.0458	.0538	.4097

TABLE 7
Trial 1

W										V*									
A	B	C	D	E	I	A	B	C	D	E	1	2	3	4	5	6	7	8	9
1	1				I	2.71	3.95	2.64	3.13	2.56	.23	.80	-.06	-.05	.31	-.06	.16	.30	.05
2	1				II	2.26	-.95	.92	-1.00	-.30	.86	.08	-.26	.07	-.06	-.10	.10	.10	.73
3	1				III	1.37	1.79	-1.80	-1.35	.27	.58	.05	.28	-.05	.25	.16	.08	.62	.58
4	1		1		IV	-.47	1.97	1.41	.47	-2.62	.68	.03	.03	-.04	.65	.30	.44	.61	.34
5					V	.50	-2.13	-.94	2.40	.09	.16	.65	-.04	-.25	-.01	.30	.07	.38	.19
6			1								.20	-.06	.31	.61	.03	-.20	-.07	.13	.72
7			1								.30	-.16	.06	.25	.01	.16	.87	.03	.08
8			1								.30	.10	.10	.73	.01	.30	.09	.16	.06
9			1		I						.05	-.08	.62	.58	.15	.05	.08	.62	.15
10					II	.38	.51	.34	.42	.35	-.02	.44	.61	.34	.19	.02	.44	.61	.34
11					III	.59	-.30	.38	-.39	-.12	-.03	.10	.36	.38	.46	.03	.10	.36	.38
12				1	IV	-.14	.56	-.77	-.29	-.94	.23	-.07	.13	.19	.72	.06	.07	.13	.19
13					V	.22	-.65	-.39	.98	-.08	-.06	.87	.03	-.08	.08	-.06	.87	.03	-.08
14	1										.83	.09	.16	.06	.06	.83	.09	.16	.06
15											.12	.16	.16	.35	.53	.12	.16	.16	.35
16			1	1		A					-.13	-.15	.01	.48	.55	-.13	-.15	.01	.48
17			1		I	.34	.42	.31	.34	.34	-.20	.35	.07	.64	.06	-.20	.35	.07	.64
18					II	.75	-.25	.34	-.31	-.12	-.03	.66	.02	-.14	.32	-.03	.66	.02	-.14
19		1		1	III	.52	.52	-.70	-.34	.13	-.03	.04	-.07	.15	.76	-.03	.04	-.07	.15
20		1			IV	-.12	.46	.42	.23	-.92	.07	.62	.59	-.06	.20	.07	.62	.59	-.06
21		1	1		V	.19	-.53	-.35	.79	-.08	.15	.06	.76	-.01	.20	.15	.06	.76	-.01
22											.04	.69	.37	.12	.12	.04	.69	.37	.12
23	1	1		1	A	1.00	.07	-.12	.17	.19	.04	.53	.56	.01	.07	.04	.53	.56	.01
24					B	.07	1.00	.06	-.27	-.14	.29	.28	.28	.13	.69	.29	.28	.28	.13
25					C	-.12	.06	1.00	.06	-.38	.16	.04	.38	.06	.28	.16	.04	.38	.06
					D	-.17	-.27	.06	1.00	-.17	.16	.04	.38	.06	.28	.16	.04	.38	.06
					E	.19	-.14	-.38	-.17	1.00	.16	.04	.38	.06	.28	.16	.04	.38	.06

* The entries in the V matrix were computed on the Matrix Multiplier (8) and may be in error by .01.

TABLE 11
Distributions of Absolute Discrepancies

<i>Discrepancy</i>	<i>Trial 1</i>	<i>Trial 2</i>	<i>Trial 2'</i>	<i>Trial 3</i>
.26	1			
.25	1			
.24	1			
.23				
.22	2			
.21				
.20	5			
.19	1			
.18	2			
.17				
.16	2			
.15	8			
.14	4			
.13	3			
.12	4	4		
.11	2			
.10	4	8		
.09	3	5	1	
.08	6	8		
.07	10	9	1	
.06	15	7	1	
.05	7	16	4	2
.04	10	19	8	5
.03	11	11	21	20
.02	10	10	22	16
.01	12	16	33	50
.00	1	12	34	32

TABLE 12
Plane B' for Trial 1

<i>W</i>		<i>B</i>		<i>V</i> *	
1		I	2.57	1	.49
2		II	.21	2	— .07
3		III	— .03	3	.25
4		IV	1.41	4	— .13
5		V	— 1.25	5	.38
6				6	.17
7		<i>L</i>		7	— .12
8				8	— .15
9		I	.33	9	.35
10	1	II	.12	10	.69
11		III	— .07	11	.13
12		IV	.43	12	— .17
13		V	— .43	13	.58
14				14	.24
15		<i>A</i>		15	.03
16				16	— .17
17		I	.47	17	.21
18		II	.17	18	.39
19		III	— .10	19	— .11
20	1	IV	.61	20	.83
21		V	— .61	21	.59
22	1			22	.70
23	1	<i>C</i>		23	.65
24				24	.07
25		A	.05	25	.55
		B'	1.00		
		C	.74		
		D	— .20		
		E	— .39		

* The entries in the V matrix were computed on the Matrix Multiplier (8) and may be in error by .01.

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